Small deviations of Gaussian processes in L_2-norm: exact asymptotics

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Abstract. Let $X(t),\ 0 \le t \le 1,$ be a centered Gaussian process. Denote $||X|| = (\int_0^1 X^2(t)dt)^{1/2}, \quad Q(X;\epsilon) = P\{||X|| \le \epsilon\}$. The problem of small deviations is to describe the behavior of $Q(X;\epsilon)$ as $\epsilon \to 0$. This problem is called also "the problem of small balls", and is known to be difficult.

Such a development is stimulated by many important mathematical problems, including the accuracy of discrete approximation for random processes, the law of the iterated logarithm in the Chung form, and the quantization problem. Small deviation theory is also related to the functional data analysis and nonparametric Bayesian estimation.

The result similar to $Q(X;\epsilon) \sim C\epsilon^{\beta}\exp(-d\epsilon^{-\alpha})$, $\epsilon \to 0$, is called the *exact* asymptotics. For instance, if $X = W$ (the Wiener process), then it is known that

$$P\{||W|| \le \epsilon\} \sim \frac{4}{\sqrt{\pi}}\ \epsilon \exp(-\frac{1}{8}\ \epsilon^{-2}),\ \epsilon \to 0.$$

The talk is dedicated to some similar results for more complicated Gaussian processes obtained in Saint-Petersburg in last years. We will discuss integrated processes, Bessel processes, Brownian excursion and meander, local Brownian time, Slepian, Bogolyubov and Ornstein-Uhlenbeck processes as well as some mixed processes.

Take as an example the exact small deviation asymptotics for the Brownian excursion $\mathfrak{e}(t), 0 \le t \le 1$. We have

$$\mathbb{P}\{\|\mathfrak{e}\| \le \varepsilon\} \sim \frac{2\sqrt{6}}{\sqrt{\pi}}\varepsilon^{-2}\exp\left(-\frac{9}{8}\varepsilon^{-2}\right), \quad \varepsilon \to 0.$$

Now denote by $\mathfrak{m}^z(t)$ the Brownian meander taking the value $z \ge 0$ at the point 1. Then it is true that for any $z \ge 0$, and as $\varepsilon \to 0$

$$\mathbb{P}\{\|\mathfrak{m}^z\| \le \varepsilon\} \sim \frac{2\sqrt{2(z^2+3)}}{\sqrt{\pi}}\varepsilon^{-2}\exp\left(-\frac{(z^2+3)^2}{8}\varepsilon^{-2} + \frac{z^2}{2}\right).$$

For $z = 0$ this result is in perfect accordance with formula (1).

We believe that the appearance of *tables of exact small deviation asymptotics* for various functionals of random processes is just a matter of time. Such tables should be similar to the tables of integrals, sums and products or to the tables of distributions of functionals of Brownian motion.