**Testing multivariate normality by zeros of the harmonic oscillator in characteristic function spaces**

NORBERT HENZE*

*KARLSRUHE INSTITUTE OF TECHNOLOGY, GERMANY

**Abstract.** We study a novel class of affine invariant tests for multivariate normality. Writing $\varphi(t) = \exp(-\|t\|^2/2)$ for the characteristic function of the $d$-variate standard normal distribution, these tests are based on the fact that the function $\varphi$ is the only solution of the partial differential equation

$$\begin{cases} \Delta\varphi(x) = (\|x\|^2 - d)\varphi(x), \quad x \in \mathbb{R}^d, \\ \varphi(0) = 1, \end{cases}$$

where $\Delta$ stands for the Laplace operator. The operator $(-\Delta + \|x\|^2 - d)$ is known as the harmonic oscillator. For $d = 1$, equation (1) reduces to a fixed point problem for the Hermite operator.

Suppose $X_1, X_2, \ldots$ is a sequence of i.i.d. $d$-variate random (column) vectors with a distribution that is assumed to be absolutely continuous with respect to Lebesgue measure. Let $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ and $S_n = n^{-1}\sum_{j=1}^n (X_j - \overline{X}_n)(X_j - \overline{X}_n)^\top$ denote the sample mean and the sample covariance matrix of $X_1, \ldots, X_n$, respectively, and write $Y_{n,j} = S_n^{-1/2}(X_j - \overline{X}_n)$, $j = 1, \ldots, n$ for the scaled residuals, where $S_n^{-1/2}$ is the symmetric square root of $S_n^{-1}$ (which exists with probability one if $n \geq d+1$). Furthermore, let

$$\varphi_n(t) = \frac{1}{n}\sum_{j=1}^n \exp\left(\mathrm{i}t^\top Y_{n,j}\right), \qquad t \in \mathbb{R}^d,$$

be the empirical characteristic function of $Y_{n,1}, \ldots, Y_{n,n}$. To test the hypothesis $H_0$ that the distribution of $X_1$ is some non-degenerate $d$-variate normal distribution, we propose the weighted $L^2$-statistic

$$T_{n,a} = n\int_{\mathbb{R}^d} \left|\Delta\varphi_n(t) - (\|t\|^2 - d)\varphi(t)\right|^2 \exp\left(-a\|t\|^2\right) w_a(t)\mathrm{d}t,$$

where $a > 0$ is a suitable tuning parameter. With the weight function $\exp\left(-a\|t\|^2\right)$, the test statistic $T_{n,a}$ allows for a simple expression that does not involve any integration and is thus amenable for computational purposes. We derive the limit null distribution of $T_{n,a}$ as $n \to \infty$ and show the consistency of a test of $H_0$ that rejects $H_0$ for large values of $T_{n,a}$ against general alternatives. As $a \to \infty$, a rescaled version of $T_{n,a}$ converges to a measure of multivariate skewness due to Móri, Rohatgi and Székely. The test shows strong empirical power with respect to prominent competitors.

Joint work with Bruno Ebner.